MATH 211/Fall 2005 FINAL EXAM

NAME:

Instructions:

a. There are thirty five questions on this exam. All of them are required.
b. Write your name on the front page of the exam as soon as you receive a copy.
c. All questions are equally weighted. The Final Exam grade accounts for 25% of the total grade of the course.
d. Justify your answers for all the problems below in order to get full credit.
e. If a wrong answer is given without any justification or explanation then no credit will be given for that question.
f. Only when the question is preceded by **T F** , answer by circling T or F when the statement given is true of false respectively.
g. Read the question well before you answer it, so as to make sure that you are answering the *same* question asked!
h. One white page is included in the end for scrap work.

(1) Suppose the variable x represents students and y represents courses and $F(x)$: x is a freshman, $B(x)$: x is full time student, $T(x, y)$: x is taking course y. Write the following statement using the above predicates and any needed quantifiers: "Every freshman is a full time student"

$\forall x \left(F(x) \rightarrow B(x) \right)$

(2) repeat the previous question with the statement: "there are two freshmen who are taking the exact same courses".

$\exists x_1 \exists x_2 \left(x_1 \neq x_2 \wedge F(x_1) \wedge F(x_2) \wedge \forall y \left(T(x_1, y) \leftrightarrow T(x_2, y) \right) \right)$

(3) Using c for "it is cold" and r for "it is rainy", write "it is rainy if it is not cold".

$\neg c \rightarrow r$

(4) Using c for "it is cold" and w for "it is windy", write "To be windy it is necessary that it be cold".

$C \longrightarrow W$

(5) What is the coefficient of $x^{11}y^7$ in the expansion of $(x+y)^{20}$?

0 because $11+7=18 \neq 20$.

(6) prove that there does not exist three consecutive odd integers (other than 3,5,7) that are all prime. (Hint: use congruence modulo an appropriate integer, then take cases)

$2n+1$, $2n+3$, $2n+5$

Divide $2n+1$ by 3. we have 3 cases:

1) $2n+1 \equiv 0 \bmod 3 \Longrightarrow 2n+1$ is not prime (as it is $\neq 3$)

2) $2n+1 \equiv 1 \bmod 3 \Longrightarrow 2n+3 \equiv 0 \bmod 3 \Rightarrow 2n+3$ is not prime

3) $2n+1 \equiv 2 \bmod 3 \Longrightarrow 2n+5 \equiv 2+4 \equiv 0 \bmod 3 \Rightarrow 2n+5$ not prime.

(7) Prove or disprove: if p and q are primes greater than 2, then $pq+1$ is never prime.

Prove True: since p & q are odd primes, pq is also odd. so $pq+1$ is even. Hence it is not prime.

(8) Use the Euclidean algorithm to find $gcd(392, 72)$

$392 = 72 \times 5 + 32$

$72 = 32 \times 2 + \underline{8}$

$32 = 8 \times 4 + 0$

so $\gcd = 8$.

(9) Use induction to prove that $1+4+7+10+\cdots+(3n-2) = \frac{n(3n-1)}{2}$

Base Case: $n=1$: $\frac{1\times(3-1)}{2} = 1$, is true.

Ind. Hypothesis: suppose, for some K, $1+4+\cdots+(3K-2) = \frac{K(3K-1)}{2}$.

Ind. step: we will prove the statement for $(K+1)$: $1+\cdots+(3K-2)+(3(K+1)-2) = \frac{K(3K-1)}{2} + 3K+1 = \frac{3K^2-K+6K+2}{2} = \frac{(K+1)(3K+2)}{2}$

$= \frac{(K+1)(3(K+1)-1)}{2}$. The statement follows by mathematical induction

(10) Define recursively the set of integers that are **not** divisible by 4. (include negative and positive values)

Base Case: $1 \in S$, $2 \in S$, $3 \in S$.

Recursive step: If $s \in S$ then $s+4 \in S$.
If $s \in S$ then $s-4 \in S$.

Use the following for questions (11) to (14) : A 5-word is any string of five letters from the English alphabet.

(11) How many 5-words are there?

26^5

(12) How many 5-words are there that contain a T? (or more)

$26^5 - 25^5$

(13) How many 5-words that begin **or** end with an A are there?

$26^4 + 26^4 - 26^3$

(14) How many 5-words have exactly one vowel?

5: to choose a vowel

21^4: to choose a 4-word of consonents.

5: ways to insert the chosen vowel:

$5 \times 5 \times 21^4$.

(15) If $S = \{1, 3, 5, 7, 11, 13\}$ then the cardinality of the power set of S is: 2^6

(16) **T** **F** Determine the truth value of the statement $\exists x \forall y P(x, y)$ where $P(x, y)$ means "$x + 2y = xy$".

(17) **T** **F** Let $A = \{x, y\}$, $B = \{x, \{x\}\}$, then $\{x\} \subset B - A$.

(18) **T** **F** For any integer n, it is always possible to find an inverse of $n - 1$ modulo n.

(19) **T** **F** It is possible to find an inverse of 12 modulo 15 while it is impossible to find an inverse of 15 modulo 12.

(20) solve the congruence equation $12x \equiv 4 \mod 17$

By the extended Euclidean Algorithm:

$5 \times 17 - 7 \times 12 = 1$, so inverse of 12 mod 17

is $-7 \equiv 10$: so $120x \equiv 40 \mod 17$

$\Rightarrow x \equiv 6 \mod 17 \Rightarrow x = 17t + 6$; $t = 0, \pm 1, \pm 2, \ldots$

(21) Give an example of a function $f : \mathbb{Z} \to \mathbb{N}$ that is onto but not one to one.

$f(z) = |z|$. one-to-one: if $z_1 \neq z_2$

then $|z_1| \neq |z_2| \Rightarrow f(z_1) \neq f(z_2)$.

onto: given $n \in \mathbb{N}$, let $z = n$. Then $f(z) = |z|$

$= |n| = n$.

(22) Write, in clear English, the negation of this statement: "I will go to the play or read a book, but not both".

"I will either not go to the play and not read a book

or I will do both".

(23) Show that n is even whenever $3n^2 + 8$ is even.

We need to show: $(3n^2+8)$ is even $\longrightarrow$ n is even

By contraposition, suppose n is odd, so $\exists$ an integer K such that $n = 2K+1$. so $3n^2+8 = 3(4K^2+4K+1)+8$ $= 12K^2+12K+11 = 2(6K^2+6K+5)+1$. This proves that $3n^2+8$ is odd.

(24) Let A and B be two sets such that $|A| = 4$ and $|B| = 7$, Find the number of one to one functions from A to B.

This is ${}_{|B|}P_{|A|} = {}_7P_4 = 7 \times 6 \times 5 \times 4$

$= 840$.

(25) Write the binomial expansion of $(2x + 1/x)^{12}$.

$$\sum_{i=0}^{12} \binom{12}{i} (2x)^i \cdot \left(\frac{1}{x}\right)^{12-i} = \sum_{i=0}^{12} \binom{12}{i} 2^i \cdot x^{2i-12}$$

Use the following for questions (26) to (29): A club with 20 women and 17 men needs to form a committee of six members.

(26) How many committees are possible?

$\binom{37}{6}$

(27) How many committees are possible if the committee must include three men and three women?

$\binom{20}{3}$ ways to choose 3 women

$\binom{17}{3}$ " " " 3 men

By the product rule we get

$$\binom{20}{3} \times \binom{17}{3} = \frac{20!}{3!\,\cancel{17!}} \times \frac{\cancel{17!}}{3!\,14!}$$

$$= \frac{20 \times 19 \times 18 \times 17 \times 16 \times 15}{6 \times 6} = 775200$$

(28) How many committees are possible if the committee must include three men and three women, **and** Angela must be on the committee? Angela is already included, so we need two more women & 3 men:
$\binom{19}{2} \times \binom{17}{3}$.

(29) The committee will consist of a president, a vice president, a secretary, a treasurer, and two members at large, AND it must consist of three men and three women, how many such committees can we make?

choose 3 women & 3 men: $\binom{20}{3} \times \binom{17}{3}$

out of the chosen 6: ${}_6P_4$ ways to choose members with titles. The remaining 2 will be automatically members at large: $\binom{20}{3} \times \binom{17}{3} \times 6\times5\times4\times3 = 279072000$

Use the following for questions (30) to (32): The n^{th} Lucas number obeys the recurrence relation $l_n = l_{n-1} + l_{n-2}$, and the initial conditions $l_1 = 2$, $l_2 = 1$.

(30) Show that $l_1^2 + l_2^2 + \cdots + l_n^2 = l_n l_{n+1} + 2$ for all $n \geq 1$.

By induction: Base Case: $n=1$: $l_1^2 = 2^2 = 4$; $l_1 l_2 + 2 = 2+2 = 4$.

Ind. hyp.: Suppose, for some integer $k \geq 1$,
$l_1^2 + \cdots + l_k^2 = l_k l_{k+1} + 2$.

Ind. Step: $l_1^2 + \cdots + l_{k+1}^2 = l_1^2 + \cdots + l_k^2 + l_{k+1}^2 = l_k l_{k+1} + 2 + l_{k+1}^2$ (by ind. hyp.) $= l_{k+1}(l_k + l_{k+1}) + 2 = l_{k+1}(l_{k+2}) + 2$

By recurrence of the sequence l_k. The proof follows by the principle of math. induction.

(31) Find the n^{th} term explicitly (i.e. solve the recurrence relation).

The characteristic equation (like that of the Fibonacci sequence) is $r^2 - r - 1 = 0$. Roots $\frac{1+\sqrt{5}}{2}$ & $\frac{1-\sqrt{5}}{2}$

general solution $\alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$.

Using $l_1 = 2$ & $l_2 = 1$ we get two equations. Solving:

we get $\alpha_1 = -\left(\frac{1-\sqrt{5}}{2}\right)$; $\alpha_2 = -\frac{1+\sqrt{5}}{2}$. Hence

$l_n = \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}$ (do it!)

(32) Find the general solution of the inhomogeneous recurrence relation: $a_n = a_{n-1} + a_{n-2} + 2^n$.

Homogeneous solution: $a_n^{(h)} = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$.

Since 2 is not a root of the characteristic eqn. we look for a particular solution of the form $C \cdot 2^n$ (compare with $p_0 s^n$, when s is not a root)
plugging this in the recurrence equation we get $C = 4 = 2^2$.

So $a_n = a_n^{(h)} + a_n^{(p)} = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n + 2^{n+2}$.

(33) Explain why the number of binary strings of length n that contain 000 as a substring can be modelled by the solution to the recurrence equation $a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$.

A binary string that has 000 starts will 1, or 01, or 001 or 000, giving

$a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$, where the last one is the number of ways to complete the remaining $(n-3)$ bits, having guaranteed that 000 is included at first. Here $n \geq 3$.

(34) Give the necessary initial conditions in the previous question.

$a_0 = 0$; $a_1 = 0$, $a_2 = 0$

(35) Write the characteristic equation for the recurrence relation $a_n = a_{n-3} + 3a_{n-4} + 2a_{n-1}$.

$r^4 - 2r^3 - r - 3 = 0$

(by assuming r^n is a solution)